§5.7 Applications to differential equations (Part II)

Suppose $x' = Ax$ for $n \times n$ matrix A. Suppose A is diagonalizable and hence has n linearly independent eigenvectors $v_1, \ldots, v_n$. Recall a set of fundamental solutions is given by

$$v_1 e^{\lambda_1 t}, \; v_2 e^{\lambda_2 t}, \; \ldots, \; v_n e^{\lambda_n t}$$

and these are the <u>eigenfunctions</u> of A. Any solution is a linear combination of these eigenfunctions.

But why?

Since $x' = Ax$ and $A = PDP^{-1}$ for $P = [v_1 | \cdots | v_n]$ and $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ we can decouple the system.

Let $y = P^{-1}x$ (or equivalently $x = Py$)

If we substitute $x = Py$ into $x' = Ax$ we obtain

(x^{ii}) $Py' = A(Py) = (PDP^{-1})Py = PDy$

$$\Rightarrow \quad Py' = PDy$$

and since P is invertible we have

$$y' = Dy$$

which is a decoupled system, which has solution $y = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$. Now multiply by P to get $x = Py$ to obtain the desired result.

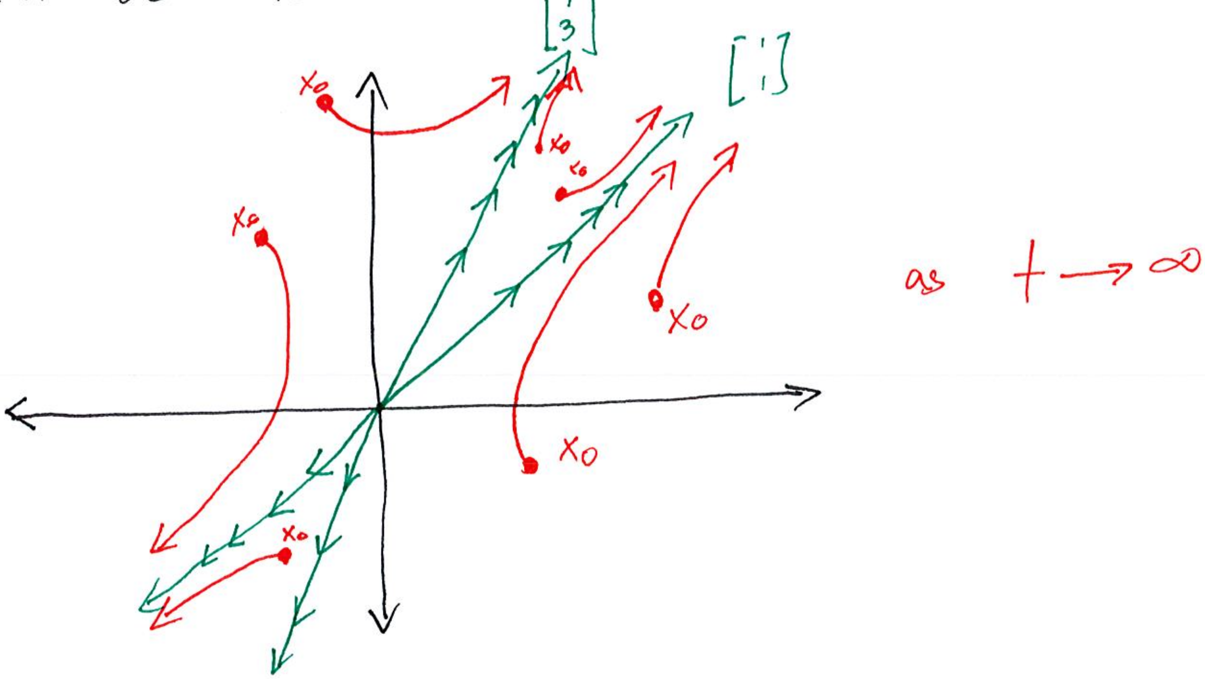
Example

Last time we saw a fundamental set of solutions to $x' = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix} x$ was

$$\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} \right\}$$

Notice both $e^{4t}, e^{6t} \longrightarrow \infty$ as $t \to \infty$. Thus we can graph <u>trajectories</u> of solutions $x(t)$.

We know $x(t) = c_1\begin{bmatrix}1\\3\end{bmatrix}e^{4t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{6t}$ and c_1, c_2 can be found with initial value $x(0) = x_0$



- In this case the origin is a <u>repeller</u> or <u>source</u> since both eigenvalues are positiv

- In the case both eigenvalues are negative, the origin is an <u>attractor</u> or <u>sink</u>

- If one eigenvalue is positive and the other negative, the origin is a <u>saddle point</u>

What if the eigenvalues are complex?

Complex Eigenvectors

Suppose $x' = Ax$ for 2×2 matrix A with complex eigenvalues. Recall if λ is an eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$. Thus

$$x_1(t) = v e^{\lambda t} \qquad x_2(t) = \overline{v}\, e^{\overline{\lambda} t}$$

is a fundamental set of solutions. Notice

$x_2(t) = \overline{x_1(t)}$ so in some sense $x_1(t)$ is enough.

Proof

To see this, it suffices to show that if $z = a+bi$, then

$$\overline{e^{z}} = e^{\overline{z}}$$

$$\overline{e^{a+bi}} = \underset{\text{real}}{\overline{e^{a}}} \cdot \overline{e^{bi}} = e^{a}\,\overline{(\cos(b) + i\sin(b))} = e^{a}(\cos(b) - i\sin(b))$$

$$= e^{a}(\cos(-b) + i\sin(-b))$$

$$= e^{a}\cdot e^{-bi}$$

$$= e^{a-bi}$$

$$= e^{\overline{a+bi}}$$

$$\begin{cases} \cos(-x) = \cos(x) \\ \sin(-x) = -\sin(x) \end{cases}$$

Since $x_2(t) = \overline{x_1(t)}$, we claim

- $\mathrm{Re}(x_1(t)) = \mathrm{Re}(ve^{\lambda t})$
- $\mathrm{Im}(x_1(t)) = \mathrm{Im}(ve^{\lambda t})$

are solutions to $x' = Ax$ and form a fundamental set of solutions. This follows since they are linearly independent and linear combinations of $x_1(t)$ and $x_2(t)$:

- $\mathrm{Re}(ve^{\lambda t}) = \frac{1}{2}\left(x_1(t) + \underbrace{\overline{x_1(t)}}_{x_2(t)}\right)$

- $\mathrm{Im}(ve^{\lambda t}) = \frac{1}{2i}\left(x_1(t) - \underbrace{\overline{x_1(t)}}_{x_2(t)}\right)$

Recall $e^{ix} = \cos(x) + i\sin(x)$

and this will produce interesting solutions to $x' = Ax$.

Example

Consider $x' = Ax$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. A has eigenvalues:

$$\lambda_1 = 1 + i \quad \text{and} \quad \lambda_2 = 1 - i$$

and eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

(verify this on your own!)

Thus we consider real and imaginary parts to $x(t) = v_1 e^{\lambda_1 t}$ (or $v_2 e^{\lambda_2 t}$, doesn't matter which)

$$v_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{t} \cdot e^{it}$$

$$= \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{t} (\cos(t) + i\sin(t))$$

$$\cancel{\mathrm{Re}(v_1 e^{\lambda_1 t}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}}$$

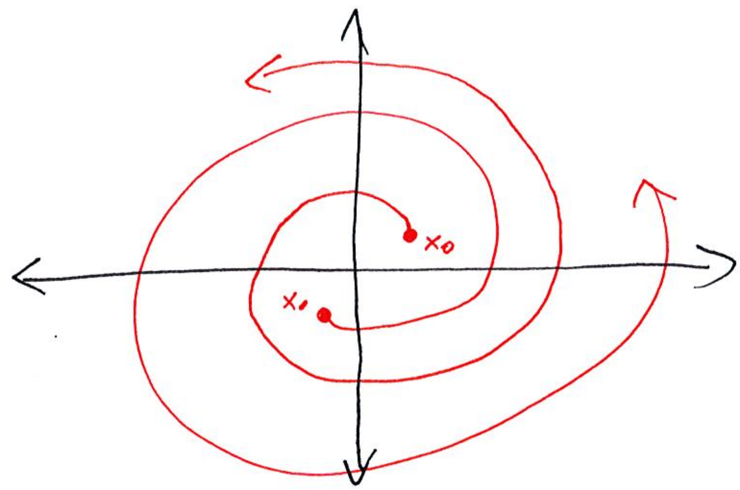
$$= e^{t} \begin{bmatrix} \cos(t) + i\sin(t) \\ -i\cos(t) + \sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} e^{t} + i \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} e^{t}$$

Thus

$$\left\{ \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} e^t , \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} e^t \right\}$$

is a fundamental set of solutions. Graphing trajectories notice $e^t \to \infty$ as $t \to \infty$ and the sines and cosines cause them ~~the~~ to "spin"



The origin is a ~~spiral source/repeller~~ spiral source / repeller in this case.

Example

Solve the initial value problem $x' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x$ and $x(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

Solution: We know $\left\{ \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} e^{t}, \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} \right\}$

is a fundamental set of solutions so

$$x(t) = c_1 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} e^{t} + c_2 \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} e^{t}$$

$$x(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -7 \end{array}\right]$$

$c_1 = 2 \qquad c_2 = -7$

$$\boxed{x(t) = 2 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} e^{t} - 7 \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} e^{t}}$$